CAPM and Methods to Determine the Market Portfolio*

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Abstract

Since 1952 modern portfolio theory developed powerful concepts that scientifically analysed financial risk. The new methods revolutionized investment and the financial industry. Finance left superstition, Voodoo and darkness. This paper explains two of these basic ideas: the *market portfolio*, a naturally selected most efficient portfolio of risky assets, and the *capital-asset-pricing model* (CAPM) which suggests how financial markets price risky assets.

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In fact almost all financial tragedies begin with a theory. *Stephen Ross*

1 Introduction and Repetition

This section describes the paper's structure and gives references to textbooks. It also highlights the principles of efficient portfolios.

1.1 Introduction

This paper summarizes the building blocks of CAPM and the methods to determine the market portfolio following the history of modern portfolio theory. To mention are mile-stones like Markowitz's concept of risk and return, Tobins's separation priniciple, or Sharpe's capital-asset-pricing model, respectively (just to name a few). Since this paper focuses on the very essentials of the concepts it does not spend much time deducing the theory in depth or even proving it but rather sketches its ideas and presents the famous results. Many details are omitted for clarity's sake and can be found in Spremann (2000). As a consequence the paper explains the concepts in a more qualitative than formal algebraic way.

At the end of most sections the reader will find *concept questions*. They point to essential parts of the concepts and test the reader's understanding. In a certain sense they are like signposts directing to aspects which are not treated in this paper. Since the questions are not always trivial, they motivate to rethink the concept and may even encourage further reading in the mentioned books (see below). In these sense the concept questions are an integral and important part of this paper. On page 13 at the end of the paper you will find a list of *key terms* used in this paper. Ideally, the reader can explain them by heart.

What are other sources of information on the topics presented in this paper? Copeland and Weston (1992) are not too formal and offer a lot of illustrative examples. The books by Varian (1992)[1] and by Mas-Collel, Whinston, and Green (1995) explain the theory from a microeconomic point of view. Once the reader is familiar with expected utility theory, corporate finance becomes even more impressive. People thirsting for a more formal approach are referred to Ingersoll (1987) or to – the more demanding – Duffie (1996). The mathematics needed is either presented in the appendices of the above mentioned books or can be learnt in Simon and Blume (1994), for example. The empirical side is covered by Campbell, Lo, and MacKinlay (1997). In order to enjoy this book the reader should have a strong background in econometrics.

Readers who prefer a more anecdotic and historical introduction may want to skim Malkiel (1990) and the two books by Bernstein (1990) and (1992). All of these three books are very entertaining.

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[1] See also Varian (1993) and Varian (1996)

For a complete reference see the bibliography at the end of the paper where a list of used abreviations and acronyms, as well as a glossary of used symbols are presented, too.

1.2 The Optimal Portfolio (A Brief Repetition)

In 1952 Markowitz[2] started Modern Portfolio Theory by defining a stock's return as a random variable. Its probability distribution is completely determined by the expected return and the return's *standard deviation*[3] (which is the square root of the variance). Since standard deviation measures the variability of the return on an individual security it is quite natural to conclude that standard deviation is an appropriate measure of the risk of an individual security. It was the first time of a quantitative description of risk. In this way Markowitz examined statistically the effect of diversification in a portfolio of securities. He found that the standard deviation of a portfolio's return is smaller than the weighted sum of its individual securities' variances. Hereby, the total impact of the diversification is determined by the correlation between these securities. Markowitz wanted to find optimally diversified portfolios by adequately weighting their individual securities. In this context the term "adequately" means "by applying mathematical algorithms".[4]

Markowitz composed diversified portfolios each containing differently weighted securities.[5] Then he identified the standard deviation and the expected return for each portfolio. All of these pairs of standard deviation and return (i.e. points) were drawn in a two-dimensional diagram with the portfolio's standard deviation as x-axis and the portfolio's return as y-axis. This graphic analysis showed that all points lied within an envelope – a hyperbola[6] opened to the right. The upper limb of this hyperbola is called *efficient frontier*. For all of its points – representing each of them a specific portfolio – are more efficient than any portfolio with the same standard deviation (i.e. a portfolio positioned vertically below); they dominate the other portfolios. On the efficient frontier a portfolio's risk can't be reduced any more without reducing its expected return.[7] In other words, they are optimally diversified. Obviously there are many of these efficient portfolios. The portfolio in the hyperbola's apex – the so-called *safety-first portfolio*[8] – is the one with minimal risk. Without other assumptions we cannot discriminate one efficient portfolio against another.

Markowitz also found that the efficient frontier can be shifted upward by increasing the number of individual securities (i.e. constructing portfolios not only out of ten individual securities but out of hundred) (see figure 1.2).

Concept Questions

1.1 What could be an investor's motivation to choose an inefficient portfolio instead of picking one on the efficient frontier?

1.2 Why are Markowitz's findings only in theoretical sense that much appealing? (What are the model's basic assumptions? What other problems did Markowitz face in 1952?)

[2]Markowitz (1952) and Markowitz (1959)

[3]The term "volatility" is synonymly used.

[4]Mathematical optimization (by defining a Lagrangian function or by using the Kuhn-Tucker approach)

[5]In the extreme a portfolio only holds one security (i.e. all weights but one are zero).

[6]It can be shown that this hyperbola is completely defined by two of its points. This theorem is the so-called *two-fund separation.*

[7]This definition is analogous to the concept of Pareto Efficiency.

[8]or *mean variance portfolio*

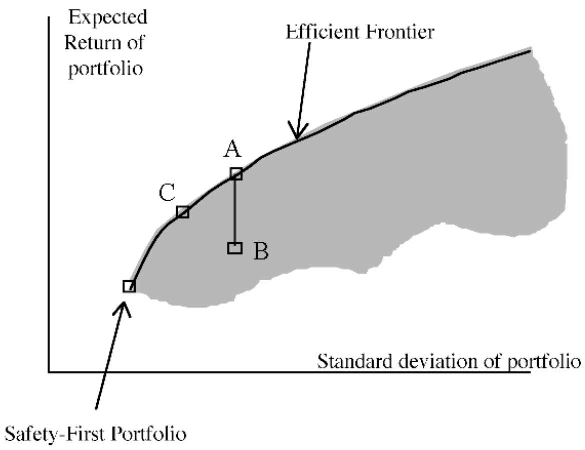

Figure 1.1: The feasible set of portfolios constructed from individual securities: The Risk-Return Diagram. The portfolios A and C are efficient, whereas B is dominated by A.

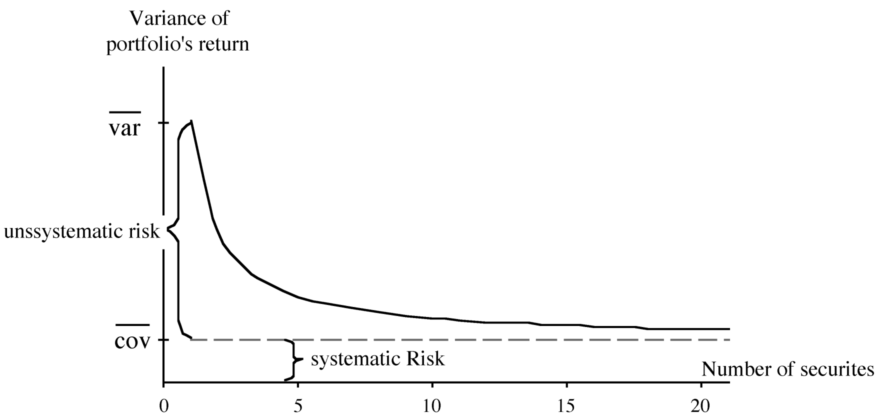

Figure 1.2: Relationsship between the variance of a portfolio's return and the number of securities in the portfolio. By the way, this graph assumes that all securities have constant variance $\overline{\text{var}}$ and constant covariance $\overline{\text{cov}}$. And finally, the securities are equally weighted. The terms "systematic risk" and "unsystematic risk" are explained in section 3.1.

The variance of the portfolio drops as more securities are added. However, it does not drop down to zero. Rather $\overline{\text{cov}}$ serves as a floor. (This figure was inspired by figure 10.7 in Ross, Westerfield, and Jaffe (1993).)

1.3 How can we further shift the efficient frontier?

1.4 When Markowitz calculated the variance of specific portfolio he had to determine the variance of each individual security and the covariances between all these securities. How many variance terms and how many covariance terms have to be calculated if the portfolio consists of N stocks? (Alternatively you could answer the question of how many diagonals exist in a convex polygon with N corners.)

1.5 Are you happy with the definition of risk in terms of standard deviation?

1.6 Why did it take such a long time before Modern Portfolio Theory arose?

2 The Market Portfolio

In this section we learn that the job of selecting the optimal portfolio and the job of choosing the appropriate risk exposure to the investor's risk appetite can be separated – just by introducing a risk-free asset.

2.1 Tobin Separation and Capital Market Line

By adding a security with risk-free returns Tobin[9] made in 1958 the next step in Modern Portfolio Theory. Now, an investor could combine any (risky) portfolio (not necessarily one from the efficient frontier) with this risk-free security (i.e. investing a part of her money in the risk-free security and the other part in any portfolio). Lets go back to the risk-return diagram (i.e. to the two-dimensional space spanned by the standard deviation of a security's return and its expected return). It is obvious that all possible portfolios consisting of a given portfolio and the risk-free security[10] lie on a straight line through the risk-free rate and the risk-return point which is characteristic for the given portfolio (see figure 1.2). From the point of view of efficiency we see that the steeper the slope of the straight line is the more efficient its portfolios become. In the extreme the straight line is tangent to the efficient frontier. This tangent is called *capital market line* (CML), and the corresponding point on the efficient frontier is the *market portfolio*. Every risky portfolio, even those on the efficient frontier, is dominated by the CML. Therefore the CML is efficient. In order to determine the market portfolio only the expected returns of all individual securities, their standard deviations and the risk-free rate have to be known. But, the market portfolio depends neither on the investors' individual preferences nor on their individual aversion towards risk. That is the reason why portfolio selection can be decomposed in (however) finding the market portfolio and in the decision of how much exposure the individual investor wants. The separation of the two jobs became known as *Tobin separation principle*.

Concept Questions

2.1 Assume the investors can borrow and lend at a risk-free interest rate. Show the points on CML for investors with different degrees of risk aversion. (How do you interpret the part of CML above the market portfolio?)

[9] Tobin (1958)

[10] A linear combination of the risk-free security and the given portfolio

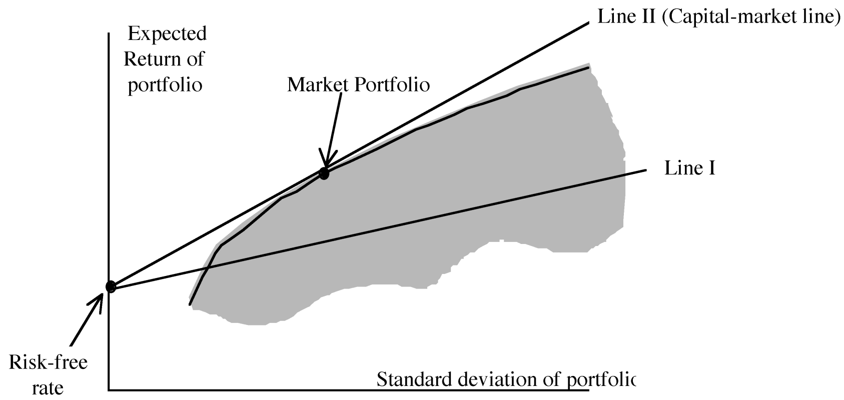

Figure 2.3: The market portfolio and the capital market line which can be viewed as the efficient set of all assets, both risky and and riskless. Line II is the capital market line. It dominates all other portfolios below.

2.2 Is there such a thing as a risk-free security? If yes, would this instrument be unique?

2.3 Why are homogeneous expectations so crucial for Tobin's separation principle?

2.4 What does Tobin's separation principle say? (voluntary: which other separation principles do you know in financial theory?)

2.2 How to Determine the Market Portfolio

In practice there are two possible ways to determine the market portfolio: It either can be calculated or it can be found by applying the Tobin separation principle.

The calculation is done in three steps. In a first step the investor has to define his IOS. The acronym IOS stands for *Investment-Opportunity-Set* and contains a universe of potential securities available for investment. This universe varies from investor to investor. The second step concerns the investor's expectations with respect to the distribution paramters of the instruments within the IOS (including all correlations). The investor has to find his relevant risk-free interest rate, as well. And, finally, a computer program, a so-called optimizer[11], calculates the weight each individual risky asset in IOS has in the market portfolio.

The *capitalization method* is an alternative way to determine the market portfolio. It uses the Tobin separation principle. Assuming that every investor has an identical IOS and homogenous expectations every investor would identically weight IOS's risky assets in the market portfolio. Therefore the capitalization of each company relative to the total market capitalization must reflect its individual weight. In order to get the relevant weights the only thing an investor has to do is gathering all the public available data on capitalization. He has outsourced the above explained optimization process to large institutional investors, so to speak.

Concept Questions

2.5 How do you determine an optimal portfolio? How do you find the market portfolio?

[11]See Spremann (2000), pp. 186-193.

2.6 If different investors have different risk-free rates does this mean that they also have different market portfolios?

2.7 How is the capitalization method interrelated with the efficient-market hypothesis? (Before answering the question, give a brief summary of the efficient-market hypothesis.)

2.8 Do bonds belong to the market portfolio?

2.3 A Market Portfolio Remains a Market Portfolio

For private investors the capitalization method is a very direct and easy way to get the weights of the market portfolio without doing a lot of research. Since this method can also be applied on classes of assets or whole industries the individual investor can replicate the market portfolios with a restricted number of individual stocks each representing such an asset class or industry. But, if an investor's IOS is too small[12] in order to represent the whole market, he possibly has to choose different weights to guarantee a still well diversified portfolio.

Concept Questions

2.9 Is a market index a good proxy for the market portfolio?

2.10 Explain why, in principle, an investor holds the market portfolio at any time once she has the market portfolio? (Was it necessary to add the term "in principle" in the last question?)

2.11 Is the market portfolio unique?

2.12 Does the market portfolio concept suggest a passive investment style?

3 Capital-Asset-Pricing Model

Above we learnt that risks can partly diversified away. Standard deviation seems not to be the ultimate measure of risk but rather a "dirty" one. For the standard deviation does not distinguish between the part of risk which can be diversified and the part of risk which is idiosyncratic. With a cleaned measure of risk (called beta) an investor can determine how much the capital market is willing to pay for a specific instrument's market risk.

3.1 The Idea of CAPM

We have seen above that in a portfolio risks can be diversified to a certain extent (see figure 1.2). That part of risk we can get rid of by diversification (for free) is called *unsystematic risk* whereas the other part is the *systematic risk*. Capital markets only pay a risk premium for systematic risk.

[12] if the IOS has not enough dimensions to build up the complete universe but only a sub-universe, so to speak.

We assume that the market portfolio is composed of N individual securities.[13] The *capital-asset-pricing model* (CAPM)[14] suggests[15] the following systematic risk for security k (with expected return μ_k and a standard deviation of σ_k, the correlation between returns on security k and the market portfolio is $\rho_{k,M}$):

$$\text{Systematic risk} = \sigma_k \cdot \rho_{k,M}$$

Hence, the expected return of security k is the risk-free interest rate i (opportunity cost, Entschädigung für aufgeschobenen Konsum) plus a premium which is proportional to the security's systematic risk:

$$\mu_k = i + p \cdot \sigma_k \cdot \rho_{k,M} \tag{3.1}$$

If σ_k is security's total risk $\sigma_k \cdot \rho_{k,M}$ is its systematic part then $\sigma_k \cdot (1 - \rho_{k,M})$ must be its unsystematic part. CAPM showed how the risk of an individual security k can be decomposed into a systematic and an unsystematic part:

$$\sigma_k = \sigma_k \cdot \rho_{k,M} + \sigma_k \cdot (1 - \rho_{k,M})$$

It can be shown that[16]

$$p = \frac{\mu_k - i}{\sigma_M} \tag{3.2}$$

Filling (3.2) into (3.1) we get after some simple algebraic reorganization the usual form of CAPM:

$$\mu_k = i + (\mu_M - i) \cdot \frac{\sigma_k \cdot \rho_{k,M}}{\sigma_M} \tag{3.3}$$

In words, the expected return for security k is equal to the risk-free interest rate plus the excess return of the market portfolio over the risk-free interest rate times the relative systematic risk. Commonly the notation β_k denotes the relative systematic risk.

$$\beta_k = \frac{\sigma_k \cdot \rho_{k,M}}{\sigma_M} \tag{3.4}$$

And hence,

$$\mu_k = i + (\mu_M - i) \cdot \beta_k \tag{3.5}$$

This equation defines a linear relationship between the beta of a security and its expected return. The resulting straight line is the so-called *security market line* (SML) (see figure 3.4, where we give also an economic explanation for this linear relationship.)

In order to prevent any misunderstanding, we want to emphasize some characteristics of CAPM:

- CAPM is a *model* in a model world with certain assumptions and restrictions. CAPM is not a law of nature!
- CAPM is an *equilibrium model*. Otherwise the argumentation in the legend of figure 3.4 would not hold.

[13]For simplicity we use the same notation as Spremann (2000)

[14]The CAPM was created by Sharpe around 1962-65 (cf. Sharpe (1964)). The version presented here is known as the *Sharpe-Lintner version of CAPM* (cf. Lintner (1965b) and (1965a)).

[15]See Spremann (2000), pp. 218-222, for a stepwise formal deduction of CAPM – as the analytical solution of an optimization problem.

[16]Skip to footnote 15. By the way, $\mu_M - i$ is the excess return of the market portfolio over the risk-free asset.

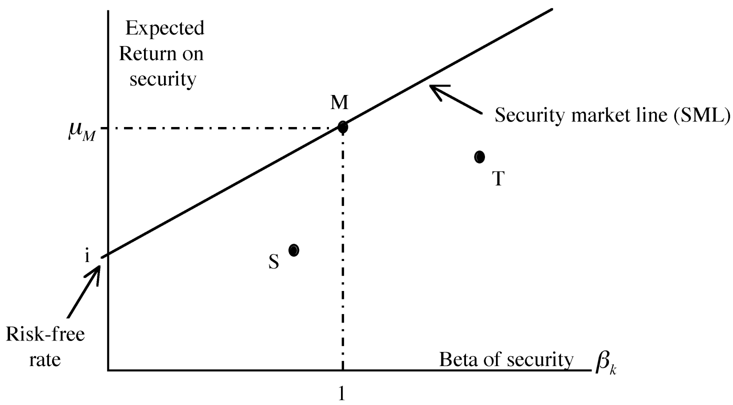

Figure 3.4: Security market line: Relationship between expected return on an individual security and the beta of the security. It is easy to see that the line in the figure is straight. To see this, consider security S with a beta smaller than 1 (e.g. 0.8). This security is represented by a point below the SML. Any investor could replicate the security S by buying a portfolio with a 20% in the risk-free asset and 80% in a security with a beta of 1 (reminds us of Tobin's separation principle). However, the replicated portfolio would itself lie on the SML (since it is a linear combination of the risk-free asset and M). In other words, the portfolio dominates the security S because the portfolio has higher expected return and the same beta. Similarly, the security T can be replicated by a portfolio on the SML which obviously dominates T.

Because no one would either hold S or T, their stock prices would drop. This price adjustment would raise the expected returns on the two securities. The price adjustment would continue until the two securities lay on the SML. This example considered two overpriced stocks and a straight SML. Securities lying above the SML are underpriced. Otherwise the price-adjustment story remains the same.

If the SML were itself curved, many stocks would be mispriced. In equilibrium, all securities would be held only when prices changed so that SML became straight. In other words, linearity would be achieved. (Source: Figure 10.11 in Ross, Westerfield, and Jaffe (1993))

- Since CAPM was deduced in this model environment CAPM is *true* (or proven, or formally correct, or without any inner contradiction). CAPM states that beta is the only determinant of the excess return on a single security. When evaluating the em-pirical relevance of CAPM this statement should be tested.
- Due to beta's prominent role, CAPM looks like a single-factor model at first glance. But the later is a regression model as multi-factor models are as well. They have no theoretical deduction but are statistical methods.

Concept Questions

3.1 What are the two components of a security's total risk? (Why doesn't diversification eliminate all risk?)

3.2 Are diversifiable risks relevant in a macroeconomic perspective?

3.3 With respect to diversification: should an employee buy shares of the company for which she works?

3.4 Which category of risk is CAPM dealing with? (Are interest rate risks diversifiable?)

3.5 Can CAPM be proven?

3.6 What are the basic assumptions the CAPM is built on? (Does the CAPM assume a (log-)normal distribution of expected returns?)

3.7 What are the differences between CML and SML?

3.8 How do you economically interpret a negative beta? What do betas of zero and one, respectively, mean?

3.9 Should an investor rationally view the variance or the beta of an individual security as the security's proper measure of risk? (Are betas and standard deviations, respectively, additive, in order to get a portfolio's total risk? Is a portfolio's standard deviation the same as its beta?)

3.10 Without having done any empirical verification of CAPM, what kind of (theoretical) weaknesses within CAPM do you suspect?

3.11 Suppose the current risk-free rate is 4 percent and the historical market risk premium is 8.4 percent. If the beta of the Cyberdairy company is 0.7, what is – using CAPM – its expected return?

3.12 The betas of VBS (Virtual Book Shelves) and of Cyberdairy are the same. The returns of VBS stocks are highly positively correlated with the domestic rate of inflation whereas those of Cyberdairy don't show such a pattern (i.e. they are not correlated). Do you expect the VBS return to differ from that of Cyberdairy? What does CAPM say?

3.13 An insurance company D acquires another insurance company F abroad. F is a publicly listed company. Nevertheless, D does no intend to take influence on F's management but rather lets them act very independently. How does an economist judge the situation with respect to diversification?

3.14 Translate the Gram-Schmidt-orthogonalization[17] method from vector geometry into the language of finance.

3.2 Applications of CAPM

There are several areas of application of CAPM. They are not explained or assessed in this paper. The reader may therefore want to glance at either Spremann (2000) or Copeland and Weston (1992).

- CAPM can be used for *valuation*. An investor can check whether a particular security is – in the sense of CAPM – over- or underpriced.
- CAPM can be applied for *corporate policy* since the model allows to determine the company's cost of equity. When the future expected return of a project is lower than a required rate the company should not go ahead with this project.
- Some *performance measures* base on CAPM.

Concept Questions

3.15 Are historical betas constant over time?

3.16
> "Bubbles are only bubbles after they burst. But discrepancies from fundamentals do contain the seeds of their own destruction. Nearly all huge, i.e., historically unprecedented, spreads against fundamentals do, indeed, eventually narrow, but when this will occur is unknown and unknowable." (Stephen Ross)

What do you think about this statement in the light of CAPM?

3.3 Empirical Tests of CAPM

CAPM was intensely tested with respect to its empirical relevance (cf. Spremann (2000) or Copeland and Weston (1992), for example). But Roll[18] doubted whether these empirical tests were sound. Possibly, CAPM cannot be tested, at all. For people often use a proxy for the market portfolio (an index like S&P500 for example). As a result one rejects CAPM's relevance just because the benchmark was not the market portfolio but an index. On the other hand, CAPM would not be reliable at all, the positive judgement only happened because of a wrong market portfolio.

Concept Question

3.17 Is Roll's critique comparable to the famous Lukas critique in macroeconomics?

[17] An algorithm to construct a minimum set of standardized orthogonal vectors spanning a space

[18] See Roll (1977).

3.4 Extensions of CAPM

CAPM relies on certain assumptions the real world does not always meet. Therefore CAPM should be extended towards a more profane environment. The list below shows some of the real world messy facts. Copeland and Weston (1992) show solutions and enhancements of CAPM in order to deal with reality.

- Lack of a riskless asset
- Non-normality: returns are not jointly normal (their distribution is rather skewed and has fat tails)
- Existence of non-marketable assets, and instruments with non-linear pay-off schemes
- Model in continuous time; different time horizons
- Heterogenous expectations, taxes, and transaction fees
- Markets are not perfectly competitive and frictionless (liquid)

Concept Question

3.18 Is APT[19] better than CAPM?

3.19 Is CAPM a one-factor model? (Is CAPM the one-dimensional version of APT?)

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[19] *Arbritrage Pricing Theory*: See Ross (1976)

A Key Terms

Beta
Capital market line
Capital-asset-pricing model
Capitalization method
Covariance and correlation
Diversification
Efficency
Efficient frontier
Homogenous expectations
Investment opportunity set
Market portfolio
Mean variance portfolio
Optimizer
Portfolio
Risk
Safety-first portfolio
Security market line
Standard deviation
Systematic risk
Unsystematic risk
Variance

B List of Used Abbreviations and Acronyms

APT	Arbitrage Pricing Theory
CAPM	Capital-Asset-Pricing Model
CML	Capital Market Line
IOS	Investment-Opportunity-Set
SML	Security Market Line

C Glossary of Used Symbols

$\overline{\text{var}}$	A positive constant
$\overline{\text{cov}}$	Another positive constant
N	A natural number
k	An index variable
M	Index for "market portfolio"
p	Proportionality factor
i	Risk-free interest rate
μ_k	Expected return of security
μ_M	Expected return of the market portfolio
σ_k	Standard deviation of security k
σ_M	Standard deviation of the market portfolio
$\rho_{k,M}$	Correlation between security k and the market portfolio M
β_k	Beta of security k

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